# Covers of Multiplicative Groups of Algebraically Closed Fields of Arbitrary Characteristic

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**Abstract**

We show that algebraic analogues of universal group covers, surjective group homomorphisms from a $\mathbb{Q}$-vector space to $F^{\times}$ with "standard kernel", are determined up to isomorphism of the algebraic structure by the characteristic and transcendence degree of $F$ and, in positive characteristic, the restriction of the cover to finite fields. This extends the main result of "Covers of the Multiplicative Group of an Algebraically Closed Field of Characteristic Zero" (B. Zilber, JLMS 2007), and our proof fills a hole in the proof given there.

## 1 Introduction

This paper was conceived as an extension of the main results of [Zil06] to fields of positive characteristic. But in the course of proving the main result a gap in the proof of the main technical theorem, Theorem 2 (in the case of $n > 1$ fields), was detected. So the aim of this paper has become twofold – to fix the proof in the characteristic zero case and to extend it to all characteristics. This goal has now been achieved.

The reader will see that we had to correct the formulation of the theorem of [Zil06]. Theorem 2.3 below now requires that the fields $L_1, \ldots, L_n$ are from an *independent system*, in the same sense as in [Zil02, Section 4], and in accordance with Shelah's theory of excellence. Indeed, the necessity of this condition has stressed again the amazingly tight interaction of field-theoretic algebra and very abstract model theory.

A simple but instructive case of Theorem 2.3 is the following statement:

Let $L_1$ and $L_2$ be linearly disjoint algebraically closed subfields of a common field of characteristic zero and $L_1 \vee L_2$ their composite. Then the multiplicative group $(L_1 \vee L_2)^{\times}$ of the composite is of the form $A \times {L_1}^{\times} \cdot {L_2}^{\times}$, for some locally free group $A$.

In characteristic $p$ the statement is true with $A$ a locally free $\mathbb{Z}[\frac{1}{p}]$-module written multiplicatively.

Here *locally free* means that any finite rank subgroup (submodule) is free.

Surprisingly, even this was apparently unknown. Our main technical proposition, Proposition 2.4, exhibits a construction which produces fields $K$ with the multiplicative group of the form $A \times D$, where $A$ is locally free and $D$ posesses

$n$-roots of elements, for any $n$. This construction is suggested by Shelah's notion of independent system and plays a crucial role in proving the uniqueness of universal covers of the multiplicative group of an algebraically closed field.

## 2 Statement of results and outline of proof

The main theorem of [Zil06] is:

**Theorem 2.1.** *For each cardinal $\kappa$ there is up to isomorphism a unique 2-sorted structure $\langle\langle V;+\rangle\,;\langle F;+,*\rangle\,;\mathrm{ex}:V\to F\rangle$ with $F$ an algebraically closed field of transcendence degree $\kappa$ such that*

$$0 \longrightarrow \mathbb{Z} \longrightarrow V \xrightarrow{\mathrm{ex}} F^{\times} \longrightarrow 1 \tag{2.1}$$

*is an exact sequence of groups.*

In positive characteristic the statement must be modified:

**Theorem 2.2.** *Given a choice of structure $\mathfrak{C}_{\mathfrak{o}} := \left\langle\langle\mathbb{Q};+\rangle\,;{\mathbb{F}_p}^{\mathrm{alg}};\mathrm{ex}_0:\mathbb{Q}\to\mu\right\rangle$, where $\mu = \left({\mathbb{F}_p}^{\mathrm{alg}}\right)^{\times}$, such that*

$$0 \longrightarrow \mathbb{Z}[\tfrac{1}{p}] \longrightarrow \mathbb{Q} \xrightarrow{\mathrm{ex}_0} \mu \longrightarrow 1\ , \tag{2.2}$$

*is an exact sequence of groups, for each cardinal $\kappa$ there is up to isomorphism a unique 2-sorted structure $\mathfrak{C} := \langle\langle V;+\rangle\,;\langle F;+,*\rangle\,;\mathrm{ex}:V\to F\rangle$ extending $\mathfrak{C}_{\mathfrak{o}}$ with $F$ an algebraically closed field of characteristic $p$ and transcendence degree $\kappa$ such that*

$$0 \longrightarrow \mathbb{Z}[\tfrac{1}{p}] \longrightarrow V \xrightarrow{\mathrm{ex}} F^{\times} \longrightarrow 1 \tag{2.3}$$

*is an exact sequence of groups.*

In fact this statement, appropriately interpreted, also makes sense in characteristic 0 and is equivalent to Theorem 2.1. Indeed we shall see that in characteristic 0 all choices of $\mathfrak{C}_{\mathfrak{o}}$ with field sort $\overline{\mathbb{Q}}$ and $\mathrm{ex}_0:\mathbb{Q}\to\mu$, where $\mu$ is the torsion group of $\overline{\mathbb{Q}}^{\times}$, are isomorphic.

Theorem 2.2 is proven by showing quasiminimal excellence ([Zil05]) of the class of models of an appropriate $L_{\omega_1,\omega}$-sentence, expressing that we have such a sequence and that ex is as specified on $\mathbb{Q}\cdot\ker(\mathrm{ex})$. For this it shall be necessary to first pass to a stronger language with a constant symbol for a generator of the kernel.

For reference, we give a quick outline of the main stages in the proof now.

Please note that for the remainder of this section we make use of terms defined only in later sections.

$p$ is zero or prime.

Quasiminimal excellency results from the truth of the following proposition, the analogue of Theorem 2 of [Zil06]

We use Shelah's notion of an *independent system*. Roughly, algebraically closed subfields $L_1,\dots,L_n$ form an independent system if they and their intersections are all linearly disjoint from each other - in the case $n=2$, the condition is precisely that $L_1$ be linearly disjoint from $L_2$ over $L_1\cap L_2$.

**Theorem 2.3.** *Let $L_1, \ldots, L_n$ be algebraically closed fields of characteristic $p$ from an independent system, subfields of some algebraically closed $F$. Let $(\overline{a}, \overline{b}) \in F^\times$ be multiplicatively independent over the product $\Pi_i {L_i}^\times$. Let $\overline{a}^{\mathbb{Q}}, \overline{b}^{\mathbb{Q}}$ be divisible subgroups of $F^\times$ containing $\overline{a}, \overline{b}$.*

*Then for some $m \in \mathbb{N}$, the field type of $\overline{b}^{\mathbb{Q}}$ over $L_1 L_2 \ldots L_n(\overline{a}^{\mathbb{Q}})$ is determined by that of $\overline{b}^{\frac{1}{m}}$ over $L_1 L_2 \ldots L_n(\overline{a}^{\mathbb{Q}})$.*

Theorem 2.3 will in turn follow by Kummer theory from the following proposition describing the structure of the multiplicative groups of finitely generated perfect extensions of composites of algebraically closed fields satisfying a certain independence condition.

By $R_p$ is meant $\mathbb{Z}[\frac{1}{p}]$ if $p > 0$ and $\mathbb{Z}$ if $p = 0$.

**Proposition 2.4.** *Let $\mathfrak{C}$ be an algebraically closed field, and let $L_1, \ldots, L_n \leq \mathfrak{C}$ be algebraically closed subfields from an independent system, $n \geq 1$. Let $K$ be the perfect closure of a finitely generated extension $L_1 \ldots L_n(\overline{\beta}) \leq \mathfrak{C}$ of $L_1 \ldots L_n$.*

*Then ${}^{K^\times}/_{\Pi_i {L_i}^\times}$ is a locally free $R_p$-module.*

Although Proposition 2.4 will suffice along with some results from [Zil06] to prove Theorem 2.2, we state here a natural extension.

**Proposition 2.5.** *In each of the following situations, ${}^{(K^{\mathrm{per}})^\times}/_H$ is a locally free $R_p$-module, where $K^{\mathrm{per}}$ is the perfect closure of $K$:*

- *$K$ is a finitely generated extension of the prime field and $H$ is the torsion group of $K^\times$*
- *$K$ is a finitely generated extension of the field generated by the group $\mu$ of all roots of unity and $H = \mu$*
- *$K$ is a finitely generated extension of the composite $L_1 \ldots L_n$ of algebraically closed fields from an independent system and $H = \Pi_i {L_i}^\times$.*

*In the first two cases, and in the third if $K$ is countable or $n = 1$, ${}^{(K^{\mathrm{per}})^\times}/_H$ is free.*

# 3 Torsion-free $R_p$-modules

**Definition 3.1.**

- For $p$ a positive prime, let $R_p$ be the subring $\mathbb{Z}[\frac{1}{p}]$ of $\mathbb{Q}$.
- For $p = 0$, let $R_p$ be the ring $\mathbb{Z}$.

To prove Theorem 2.3, we will need to work with the multiplicative groups of definably closed (i.e. perfect) subfields of $F$. These have the natural structure of $R_p$-modules. $R_p$-modules behave, even for $p > 0$, very much like Abelian groups ($\mathbb{Z}$-modules), and we borrow definitions and developments from the theory of Abelian groups.

In this section $M$ will be a torsion-free $R_p$-module written additively.

$R_p$ is a principal ideal domain with fraction field $\mathbb{Q}$, so we have the usual definitions:

**Definition 3.2.**

(i) The *span* of $A \subseteq M$, $\langle A \rangle \leq M$, is the $R_p$-submodule generated by $A$.

(ii) $\overline{a}$ is *independent* over $A \leq M$ iff

$$\forall \overline{n} \in R_p.\ (\overline{n} \cdot \overline{a} \in A \implies \overline{n} = 0).$$

$B \subseteq M$ is independent over $A$ iff every finite tuple $\overline{b} \in B$ is independent over $A$.

(iii) The *rank* of $A \leq M$, $r(A)$, is the cardinality of any maximal independent $B \in A$. This is well-defined.

(iv) $M$ is *free* of rank $\kappa$ iff it is isomorphic to the direct sum of $\kappa$ copies of $R_p$, equivalently if it is the span of an independent set (called a *basis* of $M$) of size $\kappa$.

(v) $M$ is *locally free* iff any finite rank submodule is free.

(vi) $M$ embeds in its *divisible hull* $\mathrm{divHull}(M) := M \oplus_{R_p} \mathbb{Q}$, a $\mathbb{Q}$-vector-space, and $A \leq M$ embeds in the subspace $\mathrm{divHull}(A) := A \oplus_{R_p} \mathbb{Q}$ of $M \oplus_{R_p} \mathbb{Q}$, and the embeddings commute. $R_p$-independence agrees with $\mathbb{Q}$-independence in the divisible hull, and $r(A)$ is the vector space dimension of $\mathrm{divHull}(A)$.

Our aim will be to show that certain $R_p$-modules are locally free. To this end we develop the notions of purity and simplicity:

**Definition 3.3.**

(i) A submodule $A \leq M$ is *pure* in $M$ iff $\forall a \in M.\ ((\exists n \in R_p.\ na \in A) \rightarrow a \in A)$. Equivalently, $\mathrm{divHull}(A) \cap M = A$.

(ii) The *pure hull* of $A$ in $M$, $\mathrm{pureHull}_M(A) := \mathrm{divHull}(A) \cap M$.

(iii) $\overline{a} \in M$ is *simple in $M$* iff $\overline{a}$ is independent and $\langle \overline{a} \rangle$ is pure in $M$. $\overline{a} \in M$ is *simple in $M$ mod $A$*, where $A \leq M$ is pure in $M$, iff ${}^{\overline{a}}/_{A}$ is simple in the torsion-free $R_p$-module ${}^{M}/_{A}$.

*Remark* 3.1. For $A \leq M$, the quotient $R_p$-module ${}^{M}/_{A}$ is torsion-free iff $A$ is pure in $M$.

*Remark* 3.2. In the next section we will be considering quotients of multiplicative groups of perfect fields by divisible subgroups containing the torsion. It follows from Remark 3.1 that such quotients are torsion-free $R_p$-modules.

**Lemma 3.1.** *If $A \leq M$ is pure in $M$ and $\overline{a}$ is simple in $M$ mod $A$, then $\overline{a}$ is simple in $M$.*

*Proof.* Suppose $m \cdot \alpha = a \in \langle \overline{a} \rangle$. Then ${}^{\alpha}/_{A} \in \left\langle {}^{\overline{a}}/_{A} \right\rangle$ - say $\alpha = a' + \lambda$, $\lambda \in A$, $a' \in \langle \overline{a} \rangle$.

So $m \cdot \lambda = a - m \cdot a'$. So by independence of ${}^{\overline{a}}/_{A}$, $m \cdot \lambda = 0$. Since $M$ is torsion-free, $\lambda = 0$. So $\alpha = a' \in \langle \overline{a} \rangle$. □

*Remark* 3.3. The converse holds if $A$ is divisible.

**Lemma 3.2.** *Suppose $A, B, C$ are $R_p$-modules and $B$ is an extension of $A$ by $C$:*

$$A \hookrightarrow B \xrightarrow{\phi} C \tag{3.1}$$

*Then*

1. *If $A$ and $C$ are free, then $B$ is free*
2. *If $A$ and $C$ are locally free, then $B$ is locally free*

*Proof.* 1. Say $(\phi(b_i))_{i \in I}$ is a basis for $C$. Then $(b_i)_{i \in I}$ are independent, and $B = A \oplus \langle (b_i)_{i \in I} \rangle$. So $B$ is the direct sum of free, so is free.

2. Let $B'$ be a finite rank submodule of $B$. Then we have the exact sequence:

$$A \cap B' \hookrightarrow B' \xrightarrow{\phi} \phi(B') \; . \tag{3.2}$$

But $A \cap B'$ and $\phi(B')$ are both finite rank and hence free; so B' is free by (i).

□

2 standard facts about modules over PIDs:

**Fact 3.3.** *Any finitely generated torsion-free $R_p$-module is free.*

**Fact 3.4.** *Any submodule of a free torsion-free $R_p$-module is free.*

**Lemma 3.5.** *A torsion-free $R_p$-module $M$ is locally free iff for every finite independent $\overline{a} \in M$, the pure hull of $\langle \overline{a} \rangle$ in $M$ is free.*

*Proof.* The forward direction is immediate from the definition of local freeness. For the converse, suppose $A \leq M$ is finite rank. Let $\overline{a} \in A$ be a maximal independent set. Then $A$ is contained in the pure hull of $\langle \overline{a} \rangle$, which is free by assumption. So $A$ is free by Fact 3.4. □

The next two lemmas reduce the condition of purity of a finitely generated submodule to an easily checked condition on the divisibility of points.

**Lemma 3.6.** *A finitely generated submodule $A \leq M$ is pure in $M$ iff every $a \in A$ which is simple in $A$ is simple in $M$.*

*Proof.* The forward implication is clear. Conversely, suppose $A$ is impure in $M$. Say $\alpha \in M \setminus A$, $m\alpha = a \in A$. By Fact 3.3, $A$ is free, so the pure hull of $a$ in $A$ is free of rank 1, say generated by $a'$. Then $a'$ is simple in $A$ but not in $M$. □

**Lemma 3.7.** *$a \in M$ is not simple in $M$ iff for some $\alpha \in M$ and some prime $l \neq p$, $l\alpha = a$.*

*Proof.* Suppose $a$ is not simple in $M$. Then for some $\alpha \in M \setminus \langle a \rangle$ and some $m, n \in R_p$, $m\alpha = na$. Multiplying up the equation by a power of $p$ we can take $m, n \in \mathbb{Z}$, and by changing $\alpha$ we can then take $m \notin p\mathbb{Z}$. Further we can take $gcd(m, n) = 1$. By Euclid, we have say $sm + tn = 1$. Then $m.(t\alpha + sa) = a$. Let $l$ be a prime divisor of $m$. □

We will have to deal with the delicate question of when a quotient of a locally free torsion-free $R_p$-module $M$ by a pure submodule $B$ is locally free, and more generally when for a finite tuple $\overline{c} \in M$ independent over $B$ we have that the pure hull of ${}^{\overline{c}}/_B$ in ${}^{M}/_B$ is free. It is fairly easy to see that if $B$ is finitely generated (equivalently, finite rank) then ${}^{M}/_B$ is locally free, but that the quotient by an infinite rank submodule need not be locally free.

The following lemma shows that if we find that, in a certain sense, all the "extra divisibility" of $\overline{c}$ introduced by quotienting by $B$ is explained by a finite rank portion of $B$, then the pure hull of ${}^{\overline{c}}/_B$ is indeed free.

**Lemma 3.8.** *Let $M$ be a locally free torsion-free $R_p$-module.*

*Suppose $A \leq B \leq M$, $B$ is pure in $M$, and $A$ is finitely generated.*

*Let $\overline{c} \in M$ be independent over $B$.*

*Suppose it holds for all $c \in \langle \overline{c} \rangle$ and all $m \in R_p$ that if ${}^{c}/_B$ is $m$-divisible in ${}^{M}/_B$, then already ${}^{c}/_A$ is $m$-divisible in ${}^{M}/_A$ (where $d \in D$ is said to be "$m$-divisible in $D$" iff $\exists d' \in D.\ md' = d$).*

*Then the pure hull of $\left\langle {}^{\overline{c}}/_B \right\rangle$ is free.*

*Proof.* $A = \langle \overline{a} \rangle$ say. By local freeness, the pure hull of $\langle \overline{ac} \rangle$ is free, say freely generated by $\overline{e}$.

**Claim 3.8.1.** *$\left\langle {}^{\overline{e}}/_B \right\rangle$ is the pure hull of $\left\langle {}^{\overline{c}}/_B \right\rangle$ in ${}^{M}/_B$.*

*Proof.*

- $\left\langle {}^{\overline{e}}/_B \right\rangle \leq \mathrm{divHull}(\left\langle {}^{\overline{c}}/_B \right\rangle)$: Indeed let ${}^{e}/_B \in \left\langle {}^{\overline{e}}/_B \right\rangle$. Without loss of generality, $e \in \langle \overline{e} \rangle$. $e$ is in the pure hull of $\langle \overline{ac} \rangle$, so say $s \cdot e = a + c$, $a \in \langle \overline{a} \rangle$, $c \in \langle \overline{c} \rangle$. But $A \leq B$, so $s \cdot {}^{e}/_B = {}^{c}/_B \in \left\langle {}^{\overline{c}}/_B \right\rangle$.

- $\left\langle {}^{\overline{e}}/_B \right\rangle$ is pure in ${}^{M}/_B$: Indeed suppose $m \cdot {}^{\alpha}/_B = {}^{e}/_B$, $e \in \langle \overline{e} \rangle$, $\alpha \in M$. As above, let $c \in \langle \overline{c} \rangle$ be such that $s \cdot {}^{e}/_B = {}^{c}/_B$. Then $sm \cdot {}^{\alpha}/_B = {}^{c}/_B$, so by the assumption for some $\alpha' \in M$ we have $sm \cdot {}^{\alpha'}/_A = {}^{c}/_A$. So $sm \cdot \alpha' = c + a$ say, so $\alpha' \in \langle \overline{e} \rangle$. But $sm \cdot (\alpha - \alpha') \in B$, so by purity of $B$, ${}^{\alpha}/_B = {}^{\alpha'}/_B$. So ${}^{\alpha}/_B \in \left\langle {}^{\overline{e}}/_B \right\rangle$ as required.

□

$\left\langle {}^{\overline{e}}/_B \right\rangle$ is finitely generated and so by Fact 3.3 is free. So the result follows from the Claim. □

# 4 Proof of Proposition 2.4

**Definition 4.1.** We say algebraically closed subfields $L_1, \ldots, L_n$ of an algebraically closed field $\mathfrak{C}$ are *from an independent system* iff there exist an algebraically closed subfield $C \leq \mathfrak{C}$, a finite $B \subseteq \mathfrak{C}$ algebraically independent over $C$, and subsets $B_i \subseteq B$ such that $L_i = \mathrm{acl}^{\mathfrak{C}}(C, B_i)$.

**Notation 4.2.** For subfields $F, F'$ of an algebraically closed field $\mathfrak{C}$, $F \vee F'$ is the definable closure in $\mathfrak{C}$ of $F \cup F'$, i.e. the perfect closure of the compositum of $F$ and $F'$. $F \wedge F' := F \cap F'$. $F \vee \overline{a}$ is the definable closure in $\mathfrak{C}$ of $F \cup \{a_1, ..., a_n\}$.

$\mu$ refers to the multiplicative group of all roots of unity.

We make use of some notions from valuation theory. We consider a *place* of a field $\pi : K \to k$ to be a partially defined ring homomorphism such that the domain of definition $\mathcal{O}_\pi := \mathrm{dom}(\pi)$ is a valuation ring. If $k \leq K$, we write $\pi : K \to_k k$ to indicate that $\pi$ is the identity on $k$ - in other words, that the field embedding of $k$ in $K$ is a section of $\pi$. Such a $\pi$ is sometimes called a *specialisation* of $K$ to $k$.

We make use of the Newton-Puiseux theorem and the following analogue in arbitrary characteristic:

**Fact 4.1** (Raynor [Ray68], cited in [Ked01])**.** *Let $L$ be an algebraically closed field of characteristic $p$. Let $L((t^{\mathbb{Q}}))$ be the field of generalised formal power series in $t$ with coefficients in $L$ and rational exponents, and let $L\{\{t\}\} \leq L((t^{\mathbb{Q}}))$ be the subfield consisting of those power series with support $S \subseteq \mathbb{Q}$ satisfying:*

- *there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $mS \subseteq R_p$.*

*Then $L\{\{t\}\}$ is an algebraically closed field.*

**Lemma 4.2.** *Let $L$ be an algebraically closed subfield of an algebraically closed field $\mathfrak{C}$; suppose $L$ contains algebraically closed subfields $k_i$, $i \in \{1, \dots n\}$; let $\lambda \in \mathfrak{C}$ be transcendental over $L$; let $K := \mathrm{acl}^{\mathfrak{C}}(L(\lambda)) \geq L$, and let $k_i' := \mathrm{acl}^{\mathfrak{C}}(k_i(\lambda))$. Further, let $k_0 \leq L$ be a perfect subfield, and let $k_0' := k_0$.*

*Then for any place $\pi : K \to_L L$ such that $\pi(\lambda) \subseteq \bigcap_{i>0} k_i$,*

$$\pi(\bigvee_i k_i') = \bigvee_i k_i.$$

*Proof.* Since replacing $\lambda$ with $\lambda - \pi(\lambda)$ does not alter $K$ or $k_i'$, and $\lambda - \pi(\lambda)$ is also transcendental over $L$, we may assume that $\pi(\lambda) = 0$.

Let $L\{\{\lambda\}\}$ be the field of generalised Puiseux series, as defined in Fact 4.1. Let $\pi' : L\{\{\lambda\}\} \to L$ be the standard power series residue map.

$\pi'$ agrees with $\pi$ on $L(\lambda)$, so by the Conjugation Theorem [EP05, 3.2.15] we may embed $K$ into $L\{\{\lambda\}\}$ in such a way that $\pi$ agrees with $\pi'$.

Now for $i > 0$, $k_i\{\{\lambda\}\} \leq L\{\{\lambda\}\}$, the subfield of power series with coefficients from $k_i$, is algebraically closed and contains $k_i(\lambda)$, so contains $k_i'$. Similarly, $k_0' = k_0 \leq k_0\{\{\lambda\}\}$.

Now

$$\begin{aligned} \pi(\bigvee_i k_i') &\leq \pi'(\bigvee_i (k_i\{\{\lambda\}\})) \\ &\leq \pi'((\bigvee_i k_i)\{\{\lambda\}\}) \\ &= \bigvee_i k_i \end{aligned}$$

□

**Lemma 4.3.** *Let $K \geq L$ be algebraically closed fields, and let $\pi : K \to_L L$ be a place. Let $k_0 \leq K$ be a perfect subfield such that $\pi k_0 \leq k_0$. Let $k_1 \geq k_0$ be a finite extension.*

*Then there exists a finite extension $k' \geq k_1$ such that $\pi k' \leq k'$.*

*Proof.* We may assume that $k_1/k_0$ is Galois.

For $i \geq 1$, define $k_{i+1} := k_i(\pi k_i)$.

A finite extension of a perfect field is perfect, so each $k_i$, and hence each $\pi k_i$, is perfect.

Normality of a finite field extension implies [EP05, 3.2.16(2)] normality of the corresponding extension of residue fields; it follows inductively that for all $i \geq 0$, the extensions $k_{i+1}/k_i$ and $\pi k_{i+1}/\pi k_i$ are Galois.

Now $k_{i+2}$ is generated over $k_{i+1}$ by $\pi k_{i+1}$, and $\pi k_i \leq k_{i+1}$, so $[k_{i+2} : k_{i+1}] \leq [\pi k_{i+1} : \pi k_i]$. Also, $[\pi k_{i+1} : \pi k_i] \leq [k_{i+1} : k_i]$. So after some $n$, the degrees reach their minimum level, say

$$d = [\pi k_{n+2} : \pi k_{n+1}] = [k_{n+2} : k_{n+1}] = [\pi k_{n+1} : \pi k_n] = [k_{n+1} : k_n].$$

By the fundamental inequality of valuation theory [EP05, 3.3.4],

(I) any $\sigma \in \mathrm{Gal}(k_{n+1}/k_n)$ preserves $\mathcal{O}_\pi \cap k_{n+1}$;

(II) any $\sigma \in \mathrm{Gal}(k_{n+2}/k_{n+1})$ preserves $\mathcal{O}_\pi \cap k_{n+2}$.

Now $\pi k_{n+1} = (\pi k_n)(\pi\beta)$ say, some $\beta \in k_{n+1}$. Let $\beta = \beta_1, \beta_2, \ldots, \beta_s$ be the $k_n$-conjugates of $\beta$. By (I), $\beta_i \in \mathcal{O}_\pi$ for all $i$. Applying $\pi$ to the minimum polynomial $\Pi_i(x - \beta_i)$, we see that $s = d$ and the $(\pi k_n)$-conjugates of $\pi\beta$ are precisely $(\pi\beta_i)_i$.

Now suppose for a contradiction that $\sigma \in \mathrm{Gal}(k_{n+2}/k_{n+1}) \setminus \{\mathrm{id}\}$. $k_{n+2} = k_{n+1}(\pi\beta)$, so $\sigma(\pi\beta) = \pi\beta_i$ some $i > 1$.

Now $\beta - \pi\beta \in \mathfrak{m}_\pi \cap k_{n+1}$, but $\sigma(\beta - \pi\beta) = \beta - \sigma\pi\beta = \beta - \pi\beta_i \notin \mathfrak{m}_\pi \cap k_{n+1}$. This contradicts (II).

So $d = 1$, and so $\pi k_n \leq k_n$. $\square$

**Fact 4.4.** *[May72, Proposition 1] Let $E \geq F$ be a finitely generated regular extension. Then ${E^\times}/{F^\times}$ is free as an Abelian group.*

This fact slightly extends the second statement of [Zil06, Lemma 2.1]. The proof involves considering the Weil divisors of a normal projective variety over $F$ with function field $E$.

We translate this result to our context of perfect fields and $R_p$-modules:

**Corollary 4.4.1.** *Let $E^{\mathrm{per}}$ be the perfect closure of a finitely generated regular extension $E$ of a perfect field $F$. Then ${E^{\mathrm{per}\times}}/{F^\times}$ is free as an $R_p$-module.*

*Proof.* This is immediate from Fact 4.4, on noting that if $({e_i}/{F^\times})_{i<\kappa}$ is a basis for ${E^\times}/{F^\times}$ as an Abelian group, then $({e_i}/{F^\times})_{i<\kappa}$ is a basis for ${E^{\mathrm{per}\times}}/{F^\times}$ as an $R_p$-module. $\square$

**Proposition** (2.4)**.** *Let $\mathfrak{C}$ be an algebraically closed field, and let $L_1, \ldots, L_n \leq \mathfrak{C}$ be algebraically closed subfields from an independent system, $n \geq 1$. Let $\overline{\beta} \in \mathfrak{C}$ be an arbitrary finite tuple, and let $K := L_1 \vee \ldots \vee L_n \vee \overline{\beta} \leq \mathfrak{C}$.*

*Then ${K^\times}/{\Pi_i {L_i}^\times}$ is a locally free $R_p$-module.*

*Proof.* The $n = 1$ case of Proposition 2.4 follows from Corollary 4.4.1; we proceed to prove the proposition by induction on $n$.

Let $L := L_1$, let $P := \bigvee_{i>1} L_i$, and let $H := \Pi_{i>1} {L_i}^\times \leq P^\times$.

We first show that we may reduce to the case that $\overline{\beta}$ is algebraic over $P \vee L = \bigvee_i L_i$. Indeed, the relative algebraic closure of $P \vee L$ in $P \vee L \vee \overline{\beta}$, is an algebraic subextension of the finitely generated extension $(P \vee L)(\overline{\beta})$ of $P \vee L$, and so is a finite extension $P \vee L \vee \overline{\beta}'$ say, where $\overline{\beta}' \in \mathrm{acl}^{\mathfrak{C}}(P \vee L)$.

By Corollary 4.4.1, ${}^{(P\vee L\vee\overline{\beta})^\times}/_{(P\vee L\vee\overline{\beta}')^\times}$ is free. So by Lemma 3.2, we need only show that ${}^{(P\vee L\vee\overline{\beta}')^\times}/_{HL^\times}$ is locally free.

So we suppose that $\overline{\beta} \in \mathrm{acl}^{\mathfrak{C}}(P \vee L)$.

We aim to apply Lemma 3.5. So let $\overline{b} \in P \vee L \vee \overline{\beta}$ be multiplicatively independent over $HL^\times$; we want to show that the pure hull of $\left\langle {}^{\overline{b}}/_{HL^\times} \right\rangle$ in ${}^{(P\vee L\vee\overline{\beta})^\times}/_{HL^\times}$ is free.

Let $(c_i)_i$ enumerate $\overline{\beta b}$.

**Claim 4.4.1.** *There exist a finitely generated extension $k$ of $P$ and a place $\pi : \mathrm{acl}^{\mathfrak{C}}(LP) \to_L L$ such that*

*(i)* $k \vee L \geq P \vee L \vee \overline{\beta}$;

*(ii)* $\forall i.\ c_i \in k$;

*(iii)* $L = \mathrm{acl}^L(k \cap L)$;

*(iv)* $\pi(k) = k \cap L$;

*(v)* $\pi(c_i) \in L^\times$.

*Proof.* Let $C, B, B_i$ be as in Definition 4.1. Let $\overline{\mu}$ enumerate $B \setminus B_1$.

Let $f_{i,j}(\overline{\mu}) \in L[\overline{\mu}]$ be the non-zero coefficients of a minimal polynomial in $L[\overline{\mu}][X]$ for $c_i$ over $L(\overline{\mu})$. Let $\overline{m} \in \mathrm{acl}^{\mathfrak{C}}(C)$ such that $f_{i,j}(\overline{m}) \neq 0$ for all $i, j$.

Let $\overline{a}$ be a transcendence basis for $L$ over $\mathrm{acl}^{\mathfrak{C}}(C)$. Let $Q := P \vee \overline{a}$.

We first demonstrate existence of a place $\pi : \mathrm{acl}^{\mathfrak{C}}(L(\overline{\mu})) \to_L L$ such that $\pi(\overline{\mu}) = \overline{m}$ and $\pi(Q) = Q \cap L$.

Inductively, we may assume that $\overline{\mu} = \overline{\mu}'\mu$ and $\overline{m} = \overline{m}'m$, and that $\pi' : \mathrm{acl}^{\mathfrak{C}}(L(\overline{\mu}')) \to_L L$ is a place such that $\pi'(\overline{\mu}') = \overline{m}'$ and $\pi'(Q') = Q' \cap L$, where $Q' := \overline{a} \vee \bigvee_{i>1} L_i'$, where $L_i' := \mathrm{acl}^{\mathfrak{C}}(C \cup (B_i \cap \overline{\mu}'))$.

$L_i = \mathrm{acl}^{\mathfrak{C}}(L_i'(\mu))$ if $\mu \in B_i$, else $L_i = L_i'$. Let $k_0 := \overline{a} \vee \bigvee_{\{i|i>1 \wedge \mu \notin B_i\}} L_i \leq \mathrm{acl}^{\mathfrak{C}}(L(\overline{\mu}'))$.

$\mu$ is transcendental over $L(\overline{\mu}')$, so by the Extension Theorem ([Lan72, Theorem I.1]), there exists a place $\pi'' : \mathrm{acl}^{\mathfrak{C}}(L(\overline{\mu})) \to_{\mathrm{acl}^{\mathfrak{C}}(L(\overline{\mu}'))} \mathrm{acl}^{\mathfrak{C}}(L(\overline{\mu}'))$ such that $\pi''(\mu) = m$.

By Lemma 4.2, $\pi''(Q) = Q'$.

Let $\pi := \pi' \circ \pi''$. $\pi$ is as required. By the condition on $\overline{m}$, $\pi(c_i) \in L^\times$.

Now by Lemma 4.3, there exists a finite extension $k$ of $Q(\overline{c}, \pi(\overline{c}))$ such that $\pi(k) \leq k$. $k$ and $\pi$ are as required. ∎

**Claim 4.4.2.** *If $b \in k^\times$ is simple in $k^\times$ mod $(k^\times \cap HL^\times)$, then $b$ is simple in $(k \vee L)^\times$ mod $HL^\times$.*

*Furthermore, identifying ${}^{k^\times}/_{HL^\times \cap k^\times}$ with the submodule ${}^{k^\times}/_{HL^\times}$ of ${}^{(k\vee L)^\times}/_{HL^\times}$, we have that for any $\overline{b} \in k^\times$ if $\left\langle {}^{\overline{b}}/_{HL^\times} \right\rangle$ is pure in ${}^{k^\times}/_{HL^\times}$ then it is is pure in ${}^{(k\vee L)^\times}/_{HL^\times}$.*

*Proof of Claim 4.4.2.* Suppose $b$ is not simple in $(k \vee L)^\times$ mod $HL^\times$. By Lemma 3.7 and the fact that $HL^\times$ is divisible in $(k \vee L)^\times$, we have $\alpha^q = b$ for some $\alpha \in (k \vee L)^\times \setminus k^\times$ and some prime $q \neq p$.

$k(\alpha)$ is a degree $q$ cyclic extension of $k$, so this is a Galois extension, $\mathrm{Gal}(k(\alpha)/k) \cong {}^{\mathbb{Z}}/_{q\mathbb{Z}}$, and $k(\alpha)$ is perfect.

Let $F_0 := k \cap L$ and $F_1 := k(\alpha) \cap L$. Let $F_2 \leq L$ be a finite extension of $F_1$ such that $\alpha \in k \vee F_2$ and $F_2$ is Galois over $F_0$.

By [Lan02, VI Thm 1.12], $k \vee F_2$ is Galois over $k$ and restriction to $F_2$ gives an isomorphism of finite groups

$$\restriction_{F_2} \colon \mathrm{Gal}(k \vee F_2/k) \to \mathrm{Gal}(F_2/F_0),$$

and $\mathrm{Gal}(F_2/F_1)$ is the image under $\restriction_{F_2}$ of the normal subgroup $\mathrm{Gal}(k \vee F_2/k(\alpha))$ of $\mathrm{Gal}(k \vee F_2/k)$.

So $F_1$ is Galois over $F_0$ and

$$\begin{aligned}
\mathrm{Gal}(F_1/F_0) &\cong {}^{\mathrm{Gal}(F_2/F_0)}/_{\mathrm{Gal}(F_2/F_1)} \\
&\cong {}^{\mathrm{Gal}(k \vee F_2/k)}/_{\mathrm{Gal}(k \vee F_2/k(\alpha))} \\
&\cong \mathrm{Gal}(k(\alpha)/k) \\
&\cong {}^{\mathbb{Z}}/_{q\mathbb{Z}}.
\end{aligned}$$

By [Lan02, VI Thm 1.12] again, $\mathrm{Gal}(k \vee F_1/k) \cong \mathrm{Gal}(F_1/F_0) \cong \mathrm{Gal}(k(\alpha)/k)$. So $k \vee F_1 = k(\alpha)$, and we have the following lattice diamond:

$$\begin{tikzcd}
& k(\alpha) \arrow[dl, dash, "\mathbb{Z}/q\mathbb{Z}"'] \arrow[dr, dash] & \\
k \arrow[dr, dash] & & F_1 \arrow[dl, dash, "\mathbb{Z}/q\mathbb{Z}"] \\
& k \cap L &
\end{tikzcd}$$

Since the torsion group $\mu$ is contained in $(k \cap L)^\times$, by [Lan02, VI 6.2] $F_1 = (k \cap L)(\gamma)$ for some $\gamma$ such that $\gamma^q \in k \cap L$.

Now $k(\alpha) = k \vee F_1 = k(\gamma)$, so say $\gamma = \Sigma_{i<q} c_i \alpha^i$, $c_i \in k$. Let $\sigma \in \mathrm{Gal}(k(\alpha)/k)$ restrict non-trivially to $F_1$. Say $\sigma(\alpha) = \zeta\alpha$, $\sigma(\gamma) = \zeta^l \gamma$, $(l, q) = 1$, $\zeta$ a primitive $q$th root of unity. So

$$\Sigma_{i<q} c_i \zeta^l \alpha^i = \zeta^l \gamma = \sigma(\gamma) = \Sigma_{i<q} c_i \zeta^i \alpha^i.$$

Since $(\alpha^i)_i$ is a basis for the $k$-vector-space $k(\alpha)$, we have $\gamma = c_l \alpha^l$.

Now say $sl + tq = 1$. Then $\gamma^s = c_l^s \alpha b^{-t}$. So letting $d := c_l^{-s} b^t \in k$, we have

$$d^q \gamma^{sq} = \alpha^q = b.$$

But $\gamma^{sq} = (\gamma^q)^s \in (k \cap L)^\times$, so $b$ is not simple in $k^\times$ mod $(k^\times \cap HL^\times)$.

This completes the proof of the first statement. The "Furthermore" part follows by Lemma 3.6. $\square$

We aim to apply Lemma 3.8. Let $N := {}^{k^\times}/_{H}$, which by the inductive hypothesis is a torsion-free locally free $R_p$-module; let $D := {}^{(k^\times \cap HL^\times)}/_{H}$, which is a pure submodule of $N$; and let $A := \left\langle {}^{\pi(\overline{b})}/_{H} \right\rangle \leq D$.

**Claim 4.4.3.** *Let ${}^{b}/_{H} \in \left\langle {}^{\overline{b}}/_{H} \right\rangle$, let $m \in R_p$.*

*If ${}^{b}/_{H}$ has an $m$th root modulo $D$ in $N$, then ${}^{b}/_{H}$ has an $m$th root modulo $A$ in $N$.*

*Proof.* Say ${}^{\lambda}/_{H}({}^{\alpha}/_{H})^m = {}^{b}/_{H}$, $\alpha \in k^\times$, $\lambda \in HL^\times$. $H$ is divisible, so we may suppose that $\lambda \in L^\times$, $b \in \langle \overline{b} \rangle$, and $\lambda\alpha^m = b$.

Applying $\pi$, we obtain (recalling that $\pi(b_i) \in L^\times$ and that $\pi$ fixes $L \ni \lambda$)

$$\lambda = \pi(\lambda) = {}^{\pi(b)}/_{\pi(\alpha)^m}.$$

So

$$\pi(b)\left({}^{\alpha}/_{\pi(\alpha)}\right)^m = b.$$

But $\pi(b) \in \langle \pi(\overline{b}) \rangle$ and $\pi(\alpha) \in \pi(k) \subseteq k$, so this shows that ${}^{b}/_{H}$ has an $m$th root modulo $A$ in $N$. ∎

It follows from Lemma 3.8 and Claim 4.4.3 that the pure hull of $\left\langle {}^{\overline{b}}/_{HL^\times} \right\rangle$ in ${}^{k^\times}/_{HL^\times}$ is free; by Claim 4.4.2, the pure hull in ${}^{(P \vee L \vee \overline{\beta})^\times}/_{HL^\times} = {}^{(k \vee L)^\times}/_{HL^\times}$ is also free.

Applying Lemma 3.5, this completes the proof of Proposition 2.4. ∎

**Proposition 4.5.** *In each of the following situations, ${}^{K^{\mathrm{per}\times}}/_{H}$ is a locally free $R_p$-module:*

- *$K$ is a finitely generated extension of the prime field and $H$ is the torsion group of $K^\times$*
- *$K$ is a finitely generated extension of the field generated by the group $\mu$ of all roots of unity and $H = \mu$*
- *$K$ is a finitely generated extension of the composite $L_1 \ldots L_n$ of algebraically closed fields from an independent system and $H = \Pi_i {L_i}^\times$.*

*In the first two cases, and in the third if $K$ is countable or $n = 1$, ${}^{(K^{\mathrm{per}})^\times}/_{H}$ is free.*

*Proof.* In characteristic 0, the first case is the first part of the statement of [Zil06, Lemma 2.1], and the second case is [Zil06, Lemma 2.14(ii)].

In characteristic $p > 0$, both the first and second case follow from Corollary 4.4.1 with $F$ being $K \cap {\mathbb{F}_p}^{\mathrm{alg}}$.

In all characteristics the third case is precisely Proposition 2.4. Freeness in the countable case follows from Pontyragin's theorem ([Fuc70, 19.1]), and in the $n = 1$ case from Fact 4.4. ∎

## 5 Proof of Theorem 2.3

Theorem 2.3 will follow from Proposition 2.4 by Kummer theory, our use of which is packaged in the following lemma:

**Lemma 5.1.** *Let $K$ be a perfect field containing the roots of unity $\mu$, $F \geq K$ algebraically closed. Let $\overline{a} \in K^\times$ such that ${}^{\overline{a}}/_{\mu}$ is simple in ${}^{K^\times}/_{\mu}$. Let $n \in \mathbb{N}$. Then all choices of $\overline{\alpha} \in F^\times$ such that $\overline{\alpha}^n = \overline{a}$ have the same field type over $K$.*

*Proof.* Let $\overline{\alpha}$ be such. Say $n = p^t m$, $(m,p) = 1$. Since $\overline{\alpha} \in \mathrm{dcl}(\overline{\alpha}^{p^t})$, it suffices to consider the case that $t = 0$. By Kummer theory ([Lan02, VI§8]),

$$\mathrm{Gal}(K(\overline{\alpha})/K) \cong \mathrm{Hom}\left( {}^{\langle \overline{a} \rangle_{\mathbb{Z}}}/_{\langle \overline{a} \rangle_{\mathbb{Z}} \cap (K^\times)^n}, {}^{\mathbb{Z}}/_{n\mathbb{Z}} \right) \cong {}^{\langle \overline{a} \rangle_{\mathbb{Z}}}/_{\langle \overline{a} \rangle_{\mathbb{Z}} \cap (K^\times)^n},$$

where $(K^\times)^n$ is the $n$-powers subgroup of $K^\times$.

By simplicity, $\langle \overline{a} \rangle_{\mathbb{Z}} \cap (K^\times)^n = \langle \overline{a}^n \rangle_{\mathbb{Z}}$. So $\mathrm{Gal}(K(\overline{\alpha})/K) \cong \left( {}^{\mathbb{Z}}/_{n\mathbb{Z}} \right)^{|\overline{a}|}$. □

**Notation.** Suppose we have an $n$-tuple $\overline{c} \in F^\times$ and a choice of $\overline{c}^{\mathbb{Q}}$, i.e. an agreeing choice of roots $c_i^{\frac{1}{m}}$ for each $m$ and $i$. Then for any $M = (m_{i,j})_{i,j} \in \mathrm{Mat}_{k,n}(\mathbb{Q})$, let $\overline{c}^M$ be the $k$-tuple $(\Pi_j c_j^{m_{i,j}})_i$. $\overline{c}^{\frac{1}{m}} := \overline{c}^{\mathrm{Diag}(\frac{1}{m})}$.

**Definition 5.1.** $\overline{c}^{\mathbb{Q}}$ over $K \leq F$ is *determined* by the type of $\overline{c}^M$ over $K' \leq F$ iff if $\overline{d}^{\mathbb{Q}}$ is another divisible subgroup in $F^\times$ and $\overline{d}^M$ has the same field type over $K'$ as $\overline{c}^M$, $\overline{d}^M \equiv_{K'} \overline{c}^M$, then $\overline{d}^{\mathbb{Q}} \equiv_K \overline{c}^{\mathbb{Q}}$ as long tuples - i.e. for all matrices $M'$, $\overline{d}^{M'} \equiv_K \overline{c}^{M'}$.

**Theorem** (2.3)**.** *Let $L_1, \ldots, L_n$ be algebraically closed fields of characteristic $p$ from an independent system, subfields of some algebraically closed $F$. Let $(\overline{a}, \overline{b}) \in F^\times$ be multiplicatively independent over $\Pi_i {L_i}^\times$. Let $\overline{a}^{\mathbb{Q}}, \overline{b}^{\mathbb{Q}}$ be divisible subgroups of $F^\times$ containing $\overline{a}, \overline{b}$.*

*Then for some $m \in \mathbb{N}$, the field type of $\overline{b}^{\mathbb{Q}}$ over $\overline{a}^{\mathbb{Q}} \vee \bigvee_i L_i$ is determined by that of $\overline{b}^{\frac{1}{m}}$ over $\overline{a}^{\mathbb{Q}} \vee \bigvee_i L_i$.*

*Proof.* Let $\overline{c} := \overline{a}\overline{b}$.

Let $K := \bigvee_i L_i \vee \overline{c}$. By 2.4, the pure hull of ${}^{\overline{c}}/_{\Pi_i {L_i}^\times}$ is free of rank $|\overline{c}|$, say generated by ${}^{\overline{c}'}/_{\Pi_i {L_i}^\times}$. So $\overline{c}$ and $\overline{c}'$ have the same divisible hull mod $\Pi_i {L_i}^\times$.

Since $\Pi_i {L_i}^\times$ is divisible, we may take the representatives $\overline{c}'$ such that $\overline{c}$ and $\overline{c}'$ have the same divisible hull mod $\mu$. Indeed, say $\Pi_j {c'}_j^{n_{j,k}} = c_k \cdot \lambda_k$, $\lambda_k \in \Pi_i {L_i}^\times$, $n_{j,k} \in \mathbb{Z}$. Since the rational square matrix $(n_{j,k})$ is invertible and $\Pi_i {L_i}^\times$ is divisible, we can find $\lambda'_k$ such that $\Pi_j {\lambda_j}'^{n_{j,k}} = \lambda_k^{-1}$, so multiplying $c'_j$ by $\lambda'_j$ gives the required representatives.

Furthermore, it is now clear that we can take representatives $\overline{c}' \in \overline{c}^{\mathbb{Q}} = \overline{a}^{\mathbb{Q}}\overline{b}^{\mathbb{Q}}$.

$\Pi_i {L_i}^\times$ is divisible and therefore pure in $K$, so by Lemma 3.1 ${}^{\overline{c}'}/_{\mu}$ is simple in ${}^{K}/_{\mu}$. So by Lemma 5.1, for each $n$ all choices of $\overline{c}'^{\frac{1}{n}}$ are conjugate over $K$.

So say $\overline{c}' = \overline{c}^M$, $M$ an invertible $n \times n$ matrix of rationals. Then the type of $\overline{c}^{\mathbb{Q}}$ over $K$ is determined by the type of $\overline{c}^M$ over $K$ - indeed this now follows from the fact that given $\overline{d} \in F^\times$ and $M' \in \mathrm{Mat}_{k,n}(\mathbb{Q})$, there is an $m \in \mathbb{N}$ such that any choice of $\overline{d}^{M'}$ is in the definable closure of some choice of $\overline{d}^{\left(\frac{1}{m} \cdot M\right)}$.

Now let $m \in \mathbb{N}$ large enough that $\overline{c}^M$ is in the definable closure of $\overline{c}^{\frac{1}{m}}$. Then the type of $\overline{c}^{\mathbb{Q}}$ over $K$ is determined by the type of $\overline{c}^{\frac{1}{m}}$ over $K$.

But then the type of $\overline{b}^{\mathbb{Q}}$ over $K \vee \overline{a}^{\mathbb{Q}}$ is determined by the type of $\overline{b}^{\frac{1}{m}}$ over $K \vee \overline{a}^{\frac{1}{m}}$, and so certainly by the type of $\overline{b}^{\frac{1}{m}}$ over $K \vee \overline{a}^{\mathbb{Q}}$. □

# 6 Proof of Theorem 2.2

Theorem 2.2 follows by the proof of [Zil06, Section 3], using Theorem 2.3 of the present article instead of [Zil06, Theorem 2].

However for quasi-minimality to hold, even in characteristic 0, it is necessary to pass to a slightly stronger language than that used in [Zil06]. We add to the language a constant symbol $\pi$, and replace axiom (iii) of the proof of [Zil06, Lemma 3.1] with the $L_{\omega_1,\omega}$ axiom:

$$\forall x \in \ker . \bigvee_{z \in R_p} x = z\pi.$$

In characteristic $p$ we also add axioms ensuring that the chosen $\mathfrak{C}_\mathfrak{o}$ embeds in every model, i.e. for each tuple $\overline{q}$ of rationals we add an axiom stating that $\mathrm{ex}(\overline{q}.\pi)$ satisfies the formula which isolates the field type of $\mathrm{ex}_0(\overline{q})$.

Let $\Sigma$ be the resulting axiomatisation, an $L_{\omega_1,\omega}$ sentence.

We now require for quasi-minimal excellency the following lemma:

**Lemma 6.1.** *If $H, H' \models \Sigma$ then $H$ and $H'$ satisfy the same quantifier-free sentences.*

*Proof.* We need to show that for any tuple $\overline{q}$ of rationals, $\mathrm{qftp}(\mathrm{ex}^H(\overline{q}.\pi^H)) = \mathrm{qftp}(\mathrm{ex}^{H'}(\overline{q}.\pi^{H'}))$.

In positive characteristic, this follows immediately from the extra axioms given above.

In characteristic 0, we argue as follows. The map

$$\begin{aligned} \theta : &\mu_H \to \mu_{H'} \\ &; \mathrm{ex}^H(q.\pi^H) \mapsto \mathrm{ex}^{H'}(q.\pi^{H'}) \end{aligned}$$

is a group isomorphism of the torsion groups. It follows (see [Lan02, VI 3.1]) that $\theta$ is a partial field isomorphism, as required. □

Quasi-minimal excellency of the class of models of $\Sigma$ now follows exactly as in [Zil06, Section 3], using Theorem 2.3 of the present article instead of the $n > 0$ cases of [Zil06, Theorem 2], and noting that in positive characteristic the $n = 1$ case of 2.3 proves the $n = 0$ case of [Zil06, Theorem 2].

By interpreting $\pi$ as a generator of the kernel, any structure $\mathfrak{C}$ in the statement of Theorem 2.2 gives a model of $\Sigma$. So Theorem 2.2 follows from the main theorem of [Zil05].

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